**Chapter 11**

**Infinite sequences and series**

**11.1 Infinite sequences**

**Definition:**

An infinite sequence of numbers is a function whose domain is the set of positive integers.

♦ A sequence is a list of numbers



the first term , the second term , and so on the nth term .

♦ The integer is called the index of .

♦ We can think of the sequence



as a function that sends 1 to , 2 to  and in general sends the positive integer to nth term .

♦ The sequence can be written as .

♦ The sequence



can be written



or

.

♦ The sequence  is not the same as the sequence .

**Convergence and divergence**

**Definition:**

The sequence converges to the number if for all  there exists an integer such that for all 

.

If no such number exists, we say that  diverges.

♦ If  converges to , we write



or simply , and call the limit of the sequence .

**Remark:**

If , then there exists an integer such that

.

**Example (1):**

By using the definition, prove that

1.  (2)  (3) .

**Solution:**

(1) 

Let  be given. Now we must show that there exists an integer such that for all 



, from the above remark, there exists an integer such that

 . (1)

If  (2)

then from (1) and (2) we get

.

Then .

(2) 

Let  be given. Now we must show that there exists an integer such that for all 



, there exists an integer such that

 . (3)

If  (4)

then from (3) and (4) we get

.

Then .

 (3) 

Let  be given. Now we must show that there exists an integer such that for all 



 for   (5)

If  (6)

, there exists an integer such that

 . (7)

then from (5), (6) and (7) we get

.

Then .

**Definition:** (diverges to infinity)

The sequence  diverges to infinity if for every number there exists an integer such that for all

.

If this condition holds we write

 or .

Similarly if for every number there exists an integer such that for all

,

then

 or .

**Calculating limits of sequences**

**Theorem (1):**

If ,  and is a constant, then

1. 
2. 
3. 
4. 
5. 

**Example (2):**

By using Theorem 1, find the following limit:

1. .
2. 
3. .
4. .

**Theorem (2)** (The Sandwich Theorem for sequences)

Let ,  and  be sequences of real numbers. If  holds for all and if , then

.

**Example (3):**

Prove that (by using Theorem 2)

(1)  (2)  (3) .

**Solution:**

(1) 



, then from Sandwich Theorem .

(2) 

.

, then from Sandwich Theorem .

(3) 



, then from Sandwich Theorem .

**Theorem (3):** (The Continuous Function Theorem for Sequences)

Let be a sequence of real numbers. If and if is a function that is

continuous at *L* and defined at all then .

**Example (4):**

Show that, applying Theorem 3,

1.  (2) .

**Solution:**

(1) 

Taking  and .

.

Then, by Theorem 3,



(2) 

 Taking  and .

.

Then, by Theorem 3,



**Theorem (4):**

The following six sequences converge to the limits listed below:

1. 
2. 
3. 
4. 
5.  (any )
6.  (any )

In the formulas (3) and (6), remains fixed as .

**Example (5):**

By using Theorem 4, find the following limits

(1)  (2)  (3) 

(4)  (5)  (6) 

**Solution:**

(1)  (from formula 1).

(2)  (from formula 2).

(3)  (from formula 3 with  and formula 2).

(4)  (from formula 4 with )

(5)  (from formula 5 with ).

(6)  (from formula 6 with ).

**Bounded Sequences**

**Definition:** (bounded sequence)

A sequence is called bounded if there exists a real number  such that

 for all .

**Definition:**

1. A sequence is called bounded from above if there exists a number such that

 for all .

 The number is an upper bound for .

1. A sequence is called bounded from below if there exists a number such that

 for all .

 The number is an lower bound for .

1. A sequence is called bounded if bounded from above and below.

**Example (6):**

State whether the following sequence bounded from above, bounded from below, bounded or neither ?

1.  (2) 

**Solution:**

1. The sequence  is bounded from below and lower bound is 1. This sequence is not bounded from above and so the sequence is not bounded.
2.  is bounded from below and lower bound is. Also the sequence is bounded from above because



 and has upper bound 1. Since the sequence is bounded from below and bounded

 from above, the sequence  is bounded.

**Theorem (5):**

If the sequence converges, then it is bounded.

**Increasing and Decreasing Sequences**

**Definition:**

1. A sequence is called increasing sequence (nondecreasing sequence) if

 for all .

1. A sequence is called decreasing sequence (nonincreasing sequence) if

 for all .

1. A sequence is called monotonic sequence if it is increasing or decreasing sequence.

**Example (7):**

State whether the following sequence increasing, decreasing or neither ?

1.  (2)  (3) 

**Solution:**

(1) , 

.

Then the sequence  is increasing.

(2) , 

..

Then the sequence  is increasing.

(3) 



.

Then the sequence  is decreasing.

**Theorem (6):**

An increasing sequence of real numbers converges if and only if it is bounded

from above.